

# Universal relations between nongaussian fluctuations in heavy-ion collisions

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We show that universality near a critical end point implies a characteristic relation between third- and fourth-order baryon susceptibilities  $\chi_3$  and  $\chi_4$ , resulting in a banana-shaped loop when  $\chi_4$  is plotted as a function of  $\chi_3$  along a freeze-out line. Including the individual enhancements of  $\chi_3$  and  $\chi_4$  near a critical point, these features may be a consistent set of observations supporting the interpretation of baryon fluctuations data as arising from criticality.

**Introduction**— Determining the existence of a critical point in the QCD phase diagram would provide invaluable information on the behaviour of nuclear matter at high density, indicating the existence of a first order phase transition line extending to higher density and lower temperature [1]. Models of chiral symmetry breaking suggest a critical end point (CEP) may exist at high enough temperature and low enough density to be accessible to heavy-ion collisions at small center-of-mass energy per nucleon  $\sqrt{s_{NN}}$ , motivating the beam energy scan at RHIC [2–6] and future work at FAIR. Predicting signals of a critical point in QCD is made challenging by strong-coupling in analytic theory and the sign-problem on the lattice. The motivated hypothesis of a critical point allows us to use the universality of critical phenomena to study the impact of a CEP on some observables.

One universal characteristic of a critical point is diverging correlation length  $\xi \rightarrow \infty$ , due to the order parameter field becoming massless. Higher order fluctuation moments of observables coupled to the order parameter also diverge, in particular baryon number and transverse momentum [7, 8]. These susceptibilities can be measured by event-by-event fluctuations in heavy-ion collision experiments [9]. The approach via universality requires assuming the existence of a critical point. It yields robust qualitative features which are supported by models calculations near the CEP [10–14]. However, it remains a challenge to predict quantitatively the magnitude of a signal in heavy ion collisions.

We show that a diverging correlation length implies a strict ordering of features in the third- and fourth-order baryon number susceptibilities, which offers a robust qualitative signature of criticality. Baryon number is conserved by QCD reactions, and the susceptibilities,  $\chi_n = -\partial^n \ln \mathcal{Z} / \partial \mu_B^n$ , are derivatives of the logarithm of the partition function  $\mathcal{Z}$  with respect to baryon chemical potential, are robust observables that are measured in

heavy ion collisions [15, 16]. The third-order (skewness) and fourth-order (kurtosis)

$$\chi_3 = \frac{\partial \chi_2}{\partial \mu_B}, \quad \chi_4 = \frac{\partial \chi_3}{\partial \mu_B} \quad (1)$$

are expected to provide stronger signals of criticality because they diverge with a larger power of  $\xi$  than the second order susceptibility [7]. As indicated, they are related to each other and the second-order susceptibility by a  $\mu_B$  derivative. For comparison to experimental observables, volume dependence is eliminated in the ratios

$$m_1 = \frac{T\chi_3}{\chi_2}, \quad m_2 = \frac{T^2\chi_4}{\chi_2}. \quad (2)$$

Using the 3D Ising model which is in the same universality class as QCD, we show that: (a) Universality implies that, along a freeze-out line passing near a critical end point, third and fourth-order fluctuation moments are strictly ordered

$$T_{\min, m_2} > T_{\max, m_1} > T_{\max, m_2} > T_{\text{CEP}}. \quad (3)$$

(b) Obtaining signatures of criticality does not require the experiment achieving freeze-out at temperature equal or lower than  $T_{\text{CEP}}$ . Scaling may be visible at significantly higher  $T$ , similar to the hypothesized quantum critical point in the phase diagram of high- $T_c$  superconductors which is at  $T = 0$  and masked by a superconducting regime [17]. However, the magnitudes of  $m_1$  and  $m_2$  may provide a handle on the distance from the critical point. This will be demonstrated in the NJL model with deformed couplings.

Taken together with measurements of  $\chi_3$  and  $\chi_4$  which are expected to exhibit independently particular divergences and functional dependences, this correlation of  $\chi_3$  and  $\chi_4$  along a freeze-out line is a consistency check that would support interpretation of the experimental data as bona fide signatures of criticality.

**Interplay between Skewness and Kurtosis**—We start by studying order parameter fluctuations in the 3D Ising model, which belongs to the same universality class as QCD. In the Ising model, the general coordinates are  $(H, t)$  with  $H$  the external magnetic field and

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$t \equiv (T - T_c)/T_c$  the reduced temperature. The order parameter, the magnetization  $M = -\partial \ln \mathcal{Z} / \partial H$ , is an odd function of  $H$ .  $M$  is discontinuous at  $H = 0$  for  $t < 0$ . The CEP is at the origin where the correlation length  $\xi$  diverges. The non-analytic (long-distance) behavior near the CEP is universal among systems of the same universality class while the analytic (short-distance) behavior is model dependent. The non-analytic part of the equation of state  $M = M(H, t)$  near the CEP has the scaling symmetry: if  $M$  is known for a specific positive (negative)  $t$ , then  $M$  for all positive (negative)  $t$  is known.

The susceptibilities are derivatives with respect to  $H$ ,

$$\kappa_2 \equiv \frac{\partial M}{\partial H}, \quad \kappa_3 = \frac{\partial \kappa_2}{\partial H}, \quad \kappa_4 = \frac{\partial \kappa_3}{\partial H}, \quad (4)$$

paralleling the relation of  $\chi_{n+1}$  to  $\chi_n$ , each  $\kappa_{n+1}$  is related to  $\kappa_n$  by one derivative with respect to the classical source. We first concentrate on susceptibilities at  $t = 0.2$ , obtaining susceptibilities at all  $t > 0$  by the scaling symmetry mentioned above. We expect that  $\kappa_2 = \xi^2$  is a decreasing function of  $|H|$  since  $\xi$  is larger when closer to the CEP. Also,  $\kappa_2$  is a smooth function of  $H$  since  $t = 0.2$  is in the crossover region. Therefore,  $\kappa_2$  should have a smooth peak centered at  $H = 0$  and then asymptote to zero as  $H \rightarrow \infty$  as shown in the upper panel of Fig. 1, which is computed using the non-analytic part of the equation of state of Ref. [18].

Once  $\kappa_2$  is obtained, its derivatives yield  $\kappa_3$  and  $\kappa_4$ , also shown in the upper panel of Fig. 1. It is easy to see that, for  $H < 0$ , the peak of  $\kappa_3$  coincides with  $\kappa_4 = 0$  while the peak of  $\kappa_4$  is at larger  $|H|$  than the peak of  $\kappa_3$ . Then by scaling symmetry, the peaks of  $\kappa_3$  and  $\kappa_4$  lie on lines A and B, respectively, in the lower panel of Fig. 1. At constant  $t < 0$ , however,  $\kappa_{2-4}$  are decreasing functions of  $|H|$  with no inflection points. The density plots of  $\kappa_{3(4)}$  is shown in the upper panel of Fig. 2 with a  $H$ -odd(even) heart-shaped pattern.

In QCD, the story is very similar. We expect the most singular, non-analytic behavior near the CEP has scaling symmetry as well. For a constant  $T > T_{CEP}$ ,  $\chi_2$  has the same single peak behavior as  $\kappa_2$  in the Ising model at  $t = 0.2$  shown in Fig. 1. Then  $\chi_3$  and  $\chi_4$  are obtained by taking derivatives of  $\chi_2$ , similar to analysis of the Ising model. Therefore, we expect  $\chi_3$  and  $\chi_4$  contours in the scaling region in QCD behave as depicted in the lower panel of Fig. 2 analogous to what happens in the Ising model. This feature is rather robust and is confirmed in model calculations [10–14].

### Three Scenarios in Energy Scan Experiments—

In experiments, one does not know whether the CEP exists and even if it does, what its location is *a priori*. Therefore, the question is, what can one learn from the susceptibilities measured on the freeze out line? We discuss three scenarios: in addition to the two familiar ones with or without a CEP in the phase diagram, we will also explore the possibility that a CEP is hidden at negative  $T$ .

**Scenario I: CEP at  $T > 0$** —In this scenario, con-

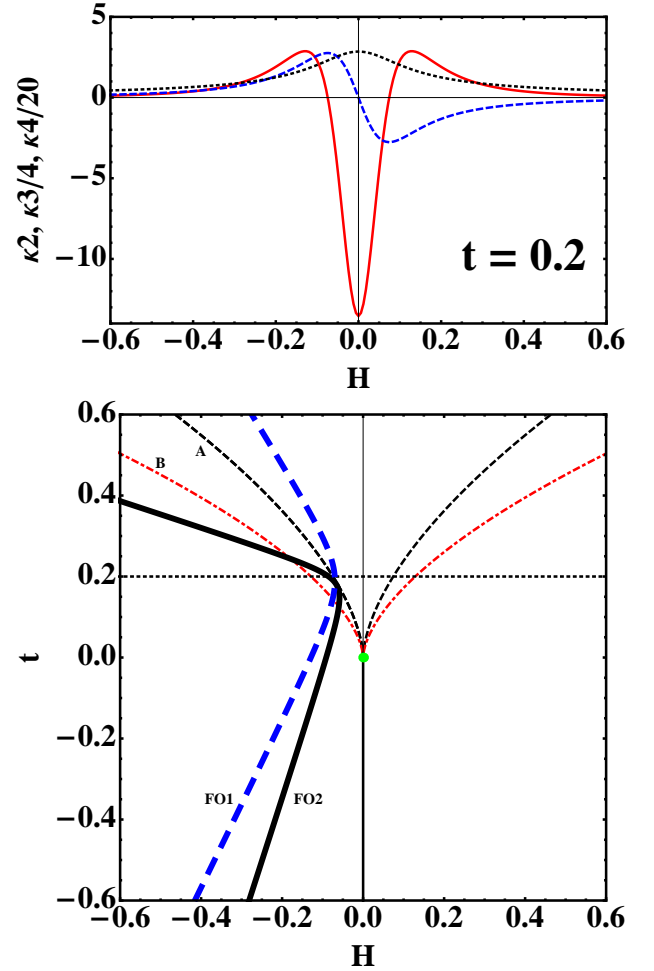


FIG. 1. Upper panel:  $\kappa_{2,3,4}(H)$  at fixed  $t > 0$ . Lower panel: the Ising model phase diagram with Line A the maximum of  $\kappa_3$  (also  $\kappa_4 = 0$ ), and Line B the maximum of  $\kappa_4$ . The curved lines are example freeze-out lines, drawn to model how they may pass through the scaling region in QCD.

tours of  $\chi_3$  and  $\chi_4$  near the CEP are as depicted in the lower panel of Fig. 2. The location of the CEP is not known precisely and neither are the contours. The boundary of  $\chi_4 = 0$  is determined by the cancellation of the leading singular structure and can be shifted by sub-leading, model-dependent, analytic contributions. Our strategy is to draw a few generic freeze out lines, depicted in the lower panels of Fig. 2 and Fig. 1, then ask whether there are common features of susceptibilities on those lines. In Fig. 1, we assume that the freeze out line is a function of  $t$ . Going from high to low  $t$ , the simplest case is FO1, which crosses lines A and B once each. The corresponding  $\kappa_4$ - $\kappa_3$  curve is shown in Fig. 3 with the curve going anti-clockwise forming a “banana” shape from high to low  $t$ . This figure shows the ordering

$$t_{\min, \kappa_4} > t_{\max, \kappa_3} > t_{\max, \kappa_4} > 0, \quad (5)$$

necessarily arises from the derivative relation between the  $\kappa_n$  and  $\kappa_{n+1}$ . All features occur at temperature higher

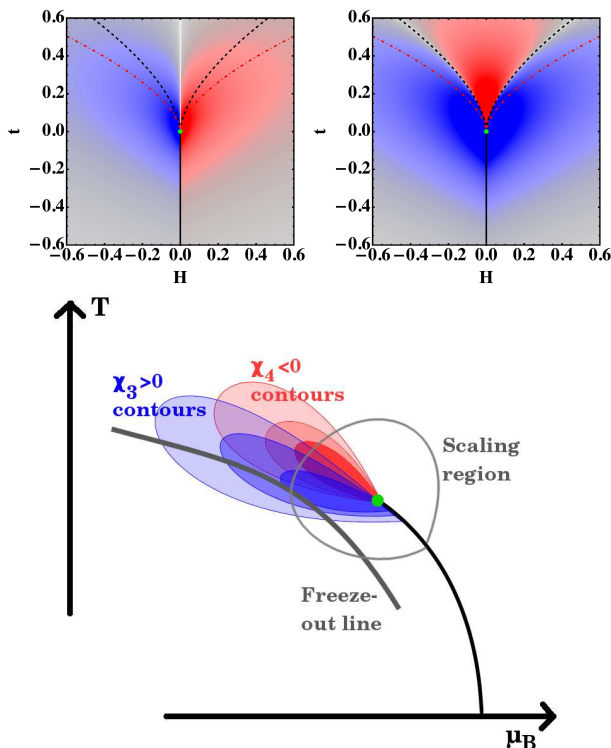


FIG. 2. Upper left (right): density plot of  $\kappa_3$  ( $\kappa_4$ ) in the Ising model. Regions of  $\kappa_i > 0$  are in blue and  $\kappa_i < 0$  are in red. The dotted (black) line is the same as Line A in Fig. 1 and dot-dashed (red) line the same as Line B. Lower panel: A sketch of the peaks in  $\chi_3$  and  $\chi_4$  on a plausible phase diagram of QCD together with a hypothetical freeze-out line. Comparison to the location of the maxima in  $\chi_3$  and  $\chi_4$  in Fig. 1 suggests how the freeze-out line may be mapped into the Ising coordinates.

than the critical point temperature. As the fluctuations become larger closer to the CEP, the closer the freeze out line to the CEP, the larger and more elongated the banana is.

In Fig. 1, we also consider a freeze out line FO2 that crosses line B twice. The corresponding  $\kappa_4$ - $\kappa_3$  plot in Fig. 3 also has the banana shape but has two local maximum peaks in  $\kappa_4$ . Those features remain when one plots  $m_2$ - $m_1$  instead of  $\kappa_4$ - $\kappa_3$  since  $\kappa_2$  changes slowly when  $\kappa_{3(4)}$  changes rapidly.

One can draw other possible freeze out lines, but the feature of an anti-clock wise loop remains, provided the line remains in the  $H < 0$  half-plane as is physically sensible for freeze-out in the hadronic phase. This can be seen from the fact that at high  $t$ , the freeze out line can start from the regime above line A, between lines A and B, or below line B, while at low  $t$ , it goes below line B. This implies these freeze out lines at high and low  $t$  will look very similar to FO1 and FO2 in Fig. 3 near the origin. This is enough to decide the loop is anti-clock wise which is a feature in common with experiment data [15, 16, 19].

**Scenario II: CEP at “ $T < 0$ ”**—As we argue above,

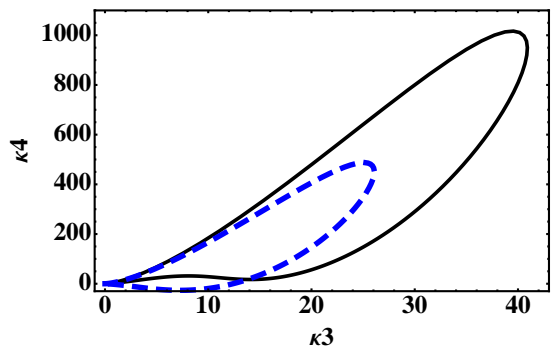


FIG. 3.  $\kappa_4$  versus  $\kappa_3$  on example freeze-out lines passing through the universal region as shown in Fig. 1. Temperature decreases in the anti-clockwise direction.

the banana shape in  $m_2$ - $m_1$  is due to the scaling symmetry governed by the CEP. But could this connection so strong such that the banana shape is observable even if the CEP is at  $T = 0$  or even  $T < 0$ ? One example is high- $T_c$  superconductors [17]. It is hypothesized that there is a quantum critical point at  $T \simeq 0$  that controls the scaling symmetry at finite  $T$  but is masked by a superconducting regime. If this is also the case in QCD, then seeing the banana shape in  $m_2$ - $m_1$  might just suggest the existence of scaling symmetry, which is likely due to a critical point, even while the critical point is at  $T < 0$  and hence technically not on the phase diagram.

To study this interesting scenario, we vary the  $K$  parameter that controls the anomaly-induced 6-fermion interaction in the Nambu–Jona-Lasinio (NJL) model [20] away from the mostly-used value  $K_0$  to reduce  $T_{\text{CEP}}$ . We find that when  $K = 0.65K_0$ ,  $T_{\text{CEP}} \simeq 0$  and when  $K = 0.4K_0$ ,  $T_{\text{CEP}}$  effectively becomes negative by extrapolation. The phase boundaries for  $K = 1.0, 0.65, 0.4K_0$  are shown in the insets in Figures 4, 5 and 6, respectively, together with the hypothetical freeze out lines which are obtained by rescaling  $\mu_B$  of the phase boundaries by factors of 0.98, 0.95, 0.915 for the red, green, blue curves, respectively. We have also plotted the corresponding  $m_2$ - $m_1$  plots in Figs. 4-6 which exhibit the following features:

(1) If freeze-out can occur at or below  $T_{\text{CEP}}$ , all three features are visible: the minimum in  $m_2$ , maximum in  $m_1$  and maximum in  $m_2$ . However, if  $T_{\text{CEP}}$  is too low to be reached by the experimental conditions, only a subset of these features appears in the data. Which subset provides a rough guide to how far away  $T_{\text{CEP}}$  is, as well as an estimation of the upper limit of  $T_{\text{CEP}}$ .

(2) The magnitude of the susceptibilities also changes significantly with the distance from a critical point. This is because the dimensionless  $m_{1(2)}$  scales with positive powers of  $\xi$  divided by typical scales in the system set by  $\mu_B$  and  $T$ , and  $m_2$  scales with more powers of  $\xi$  than  $m_1$ . Therefore when the freeze out line is far away from the CEP, both  $m_1$  and  $m_2$  are limited to small values  $\lesssim 1$  and  $m_2$  is the same magnitude as  $m_1$ . Nearer a critical point,  $m_2$  achieves values significantly greater than 1.

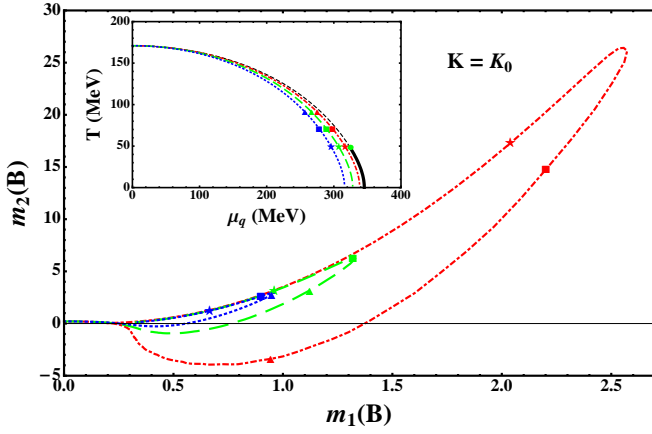


FIG. 4. Inset: the phase diagram of the NJL model with  $K = K_0$ , a value chosen to reproduce QCD observables, and three hypothetical freeze-out lines tracking the phase boundary (see text). Larger frame:  $m_2$  versus  $m_1$  on the freeze-out lines plotted in the inset.

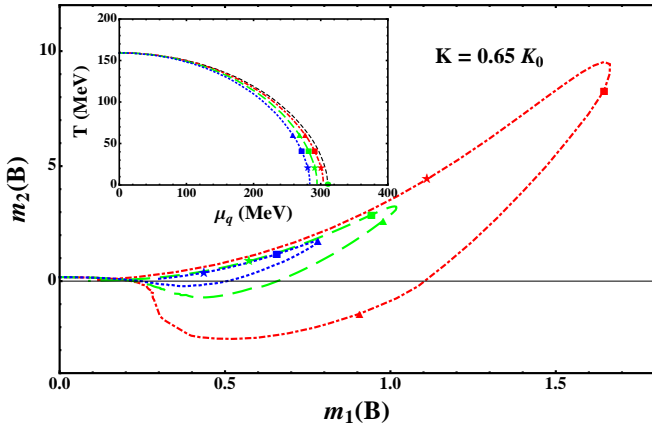


FIG. 5. Same as Fig. 4 but with  $K = 0.65K_0$ . The critical end point is at  $T_{\text{CEP}} = 0$ .

This is the reason  $m_1$ - $m_2$  banana plots become larger and more elongated for freeze-out lines in closer proximity to a critical point.

**Scenario III: No CEP at all**—We expect no  $\xi$  enhancement for  $m_{1(2)}$ , and no scaling symmetry associated with the banana shape in the  $m_2$ - $m_1$  plot.

**Summary**—Putting these observations together, we may formulate rough criteria for evidence of proximity to a critical point. If  $m_2$  achieves maximum  $\gtrsim 2$ , accompanied by  $m_1$  maximum  $\gtrsim 1$ , and the maxima obey the ordering Eq. (3), then freeze-out is occurring near enough to a critical point for universality to provide the leading order dynamics.

Preliminary data for  $m_1$  and  $m_2$  [19] suggest that the lowest collision energies may be entering the scaling region, seeing that  $m_2/m_1 \sim 4 \gg 1$ , but not reaching as

low as  $T_{\text{CEP}}$ , seeing that the banana shape is not complete.

There are of course experimental challenges associated with measuring  $\chi_3$  and  $\chi_4$ , such as effects from dynamics

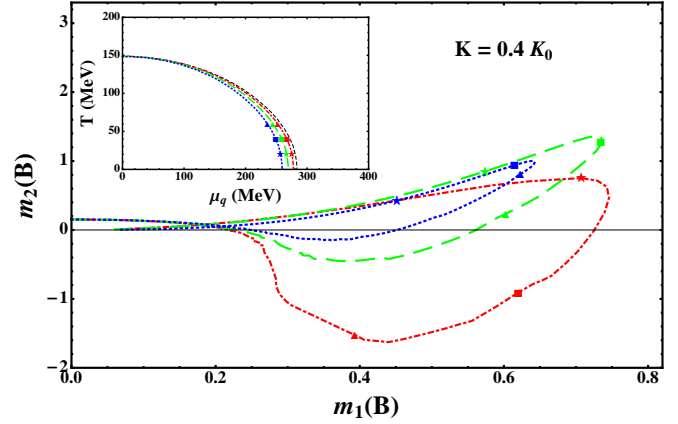


FIG. 6. Same as Figure 5 with  $K = 0.4K_0$ . The critical end point would be formally at  $T_{\text{CEP}} < 0$ . Note the difference in scale of the axes.

of the fireball expansion [21] and beam energy-dependent final-state hadron-gas interactions. These effects must be understood and accounted for in the data as well as possible to enhance any signal of criticality. Our purpose has been to argue that, if freeze-out occurs near enough to a critical point (in the scaling region) for nongaussian fluctuations to be impacted as predicted in preceding work, then the qualitative relation between the higher-order fluctuation moments Eq. (3) could hold. This can serve as a first check that the experimentally observed peaks in  $m_1$  and  $m_2$  arise from an underlying divergence of the correlation length in the fireball. We anticipate that several more such consistent sets of characteristics should be found in order to prove that the experimental data do reflect the presence of a critical point.

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